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New explicit solutions of the Klein–Gordon equation using the variational iteration method combined with the Exp-function method

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A R T I C L E I N F O

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A B S T R A C T

One of the advantages of the variational iteration method is the free choice of initial guesses. In this paper we use the basic idea of the Exp-function method to construct a generalized trial function with some unknown parameters. The Klein–Gordon equation is used to illustrate the effectiveness and convenience of the method, some new explicit travelling wave solutions are obtained which include bell-type soliton solution, kink-type soliton solution, and generality solitary wave solutions.

1. Introduction

In recent years, several powerful methods have been proposed to obtain exact solutions of nonlinear partial differential equations (PDEs), such as the tanh-function method [1], the homotopy perturbation method [2–5], the variational iteration method [6–8], and the Exp-function method [9,10]. In recent years, the direct search for exact solutions of PDEs has become more and more attractive partly due to the availability of computer symbolic systems like Maple or Mathematica, which allows us to perform the complicated and tedious algebraic calculations on computer. In particular, one of the most effective direct methods to construct exact solutions of PDEs is the Exp-function method. The Exp-function method can be used to seek solitary solutions, periodic solutions and compacton-like solutions of nonlinear differential equations.

In this paper, we will apply the basic idea of the Exp-function method to obtain the explicit exact solutions using the variational iteration method. Some new solitary solutions are obtained for the Klein–Gordon equation.

2. The variational iteration method

To illustrate the basic concepts of variational iteration method, we consider the following differential equation

$$Lu + Nu = g(x), \tag{1}$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is an inhomogeneous term.

According to the variational iteration method, we can construct a correct functional as follows

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda[Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)]\mathrm{d}\tau, \tag{2}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, the subscript n denotes the nth-order approximation, is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$.

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One of the advantages of the variational iteration method is the free choice of the initial solution, u_0. Hinted by He's Exp-function method [11], we assume that the initial solution can be expressed in a generalized form:

$$u_0(x,t) = \frac{\sum_{n=-e}^{d} a_n \exp(n\xi(x,t))}{\sum_{m=-l}^{q} b_m \exp(m\xi(x,t))}, \tag{3}$$

where e, d, l and q are positive integers which are unknown to be further determined, a_n, b_m are unknown constants, and is a function of (x, t). Eq. (3) can be re-written in an alternative form as follows:

$$u_0(x,t) = \frac{a_e \exp(e\xi) + \cdots + a_{-d} \exp(-\mathrm{d}\xi)}{b_l \exp(l\xi) + \cdots + b_{-q} \exp(-q\xi)}. \tag{4}$$

In order to determine values of e, l, q and d, we balance the linear term of highest order in Eq. (1) with the highest-order nonlinear term.

In order to identify the constants in the initial solution, we can set

$$u_n(x,t) = u_{n+1}(x,t) \tag{5}$$

and

$$\frac{\partial^k}{\partial t^k} u_n(x,t) = \frac{\partial^k}{\partial t^k} u_{n+1}(x,t). \tag{6}$$

From Eqs. (5) and (6), we obtain a set of algebraic polynomials for $\exp(j\xi)$ $(j = 0, 1, 2 \ldots)$. Eliminating all the coefficients of the powers of $\exp(j\xi)$ yields a series of algebraic equations, from which the parameters a_n, b_m, and ξ are explicitly determined.

Finally, substituting a_n, b_m, and ξ obtained in the above into (4), we can derive the exact solutions of Eq. (1).

To illustrate the effectiveness and convenience, we consider in the next section the Klein–Gordon equation as an example.

3. The explicit exact solutions of the Klein–Gordon equation

Now, we consider the Klein–Gordon equation [12]

$$u_{tt} - u_{xx} + \alpha u - \beta u^3 = 0, \tag{7}$$

where α, β are constants. This equation is the most natural nonlinear generalization of the wave equation, and appears in many different fields of application. For instance, a polynomial nonlinearity can be used as a model field theory, while a $\cos(u)$ term yields the sine-Gordon equation and so on.

To solve Eq. (7) by means of the proposed method, we construct a correction functional which reads as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(u_{ntt} - \tilde{u}_{nxx} + \alpha\tilde{u}_n - \beta\tilde{u}_n^3)\mathrm{d}\tau, \tag{8}$$

where $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$. Its stationary conditions can be obtained as follows:

$$\lambda''(\tau) = 0, \qquad 1 - \lambda'(\tau)|_{\tau=t} = 0, \quad \lambda(\tau)|_{\tau=t} = 0. \tag{9}$$

The Lagrange multiplier, therefore, can be obtained as $\lambda = \tau - t$, and the following variational iteration formula can be obtained as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\tau - t)(u_{n\tau\tau} - u_{xx} + \alpha u_n - \beta u_n^3)\mathrm{d}\tau. \tag{10}$$

To search for its exact travelling wave solution, we assume the initial solution of the form

$$u_0(\xi) = \frac{a_e \exp(e\xi) + \cdots + a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi) + \cdots + a_{-d}\exp(-\mathrm{d}\xi)}{b_l \exp(l\xi) + \cdots + b_1 \exp(\xi) + b_0 + b_{-1}\exp(-\xi) + \cdots + b_{-q}\exp(-q\xi)}, \tag{11}$$

where $a0, a1, \ldots, ae, , a-1, \ldots, a-d, b0, b1, \ldots, bl, b-1, \ldots, b-q$ are constants to be determined. $\xi = kx + wt$. By balancing the highest-order derivative term with the nonlinear term in Eq. (7), we obtain $e = l = 1$, $q = d = 1$ in Eq. (11).

$$u_0(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1}\exp(-\xi)}. \tag{12}$$

With the aid of Maple, substituting Eq. (12) into Eq. (10), we can calculate $u1$ and $u2$. In order to identify the constants in the initial solution, we can set

$$u_n(x,t) = u_{n+1}(x,t), \tag{13}$$

and

$$\frac{\partial^k}{\partial t^k}u_n(x,t) = \frac{\partial^k}{\partial t^k}u_{n+1}(x,t). \tag{14}$$

Using Maple, we have

$$\frac{\partial}{\partial t}u_0(x,t) = \frac{w[(a_1b_0 - a_0b_1)e^{(kx+wt)} + (a_0b_{-1} - a_{-1}b_0)e^{-(kx+wt)} + 2(a_1b_{-1} - a_{-1}b_1)]}{(b_1e^{(kx+wt)} + b_0 + b_{-1}e^{-(kx+wt)})^2}. \tag{15}$$

Similarly we can obtain explicitly the expression for $\frac{\partial}{\partial t}u_1(x,t)$. Setting $\frac{\partial}{\partial t}u_0(x,t) = \frac{\partial}{\partial t}u_1(x,t)$, and equating the coefficients of $\exp[n(kx+wt)](n = -3, -2, -1, 0, 1, 2, 3)$ yields

$$\begin{aligned}&3w^2a_{-1}b_0b_1 - \beta a_0^3 + 2\alpha a_0b_1b_{-1} + 2\alpha a_{-1}b_1b_0 + \alpha a_0b_0^2 - 6\beta a_1a_0a_{-1} + 3w^2a_1b_0b_{-1}\\&\quad - 6w^2a_0b_1b_{-1} + 6a_0b_1b_{-1}k^2 + 2\alpha a_1b_{-1}b_0 - 3k^2a_{-1}b_0b_1 - 3k^2a_1b_{-1}b_0 = 0,\end{aligned} \tag{16}$$

$$\begin{aligned}&6\beta a_0a_1a_{-1} - \alpha a_0b_0^2 - 2\alpha a_1b_0b_{-1} + \beta a_0^3 - 2\alpha a_0b_1b_{-1} + 3k^2a_1b_0b_{-1} - 2\alpha a_{-1}b_1b_0\\&\quad + 6w^2a_0b_1b_{-1} - 6k^2a_0b_1b_{-1} + 3k^2a_{-1}b_1b_0 - 3w^2a_1b_0b_{-1} - 3w^2a_{-1}b_1b_0 = 0,\end{aligned} \tag{17}$$

$$\beta a_1^3 - \alpha a_1b_1^2 = 0, \tag{18}$$

$$\beta a_1^3 - \alpha a_{-1}b_{-1}^2 = 0, \tag{19}$$

$$\alpha a_1b_1^{2-}\beta a_1^3 = 0, \tag{20}$$

$$\begin{aligned}&4w^2a_{-1}b_1^2 - 4k^2a_{-1}b_1^2 + \alpha a_1b_0^2 + w^2a_1b_0^2 - 3\beta a_{-1}a_1^2 - 3\beta a_1a_0^2 - 4w^2a_1b_{-1}b_1 - w^2a_0b_0b_1\\&\quad + 2\alpha a_0b_0b_1 - k^2a_1b_0^2 + k^2a_0b_0b_1 + 4k^2a_1b_{-1}b_1 + 2\alpha a_1b_{-1}b_1 + \alpha a_{-1}b_1^2 = 0,\end{aligned} \tag{21}$$

$$k^2a_0b_{-1}^2 - w^2a_0b_{-1}^2 - \alpha a_0b_{-1}^2 + w^2a_{-1}b_0b_{-1} - 2\alpha a_{-1}b_0b_{-1} + 3\beta a_0a_{-1}^2 - k^2a_{-1}b_{-1}b_0 = 0, \tag{22}$$

$$w^2a_1b_1b_0 + 3\beta a_0a_1^2 - \alpha a_0b_1^2 - w^2a_0b_1^2 - 2\alpha a_1b_1b_0 + k^2a_0b_1^2 - k^2a_1b_0b_1 = 0, \tag{23}$$

$$\begin{aligned}&4k^2a_{-1}b_1b_{-1} - 4w^2a_{-1}b_1b_{-1} - w^2a_0b_0b_{-1} + 2\alpha a_{-1}b_1b_{-1} + \alpha a_1b_{-1}^2 + 2\alpha a_0b_0b_{-1} + \alpha a_{-1}b_0^2\\&\quad + 4w^2a_1b_{-1}^2 - 3\beta a_0^2a_{-1} - 4k^2a_1b_{-1}^2 + k^2a_0b_0b_{-1} + w^2a_{-1}b_0^2 - 3\beta a_1a_{-1}^2 - k^2a_{-1}b_0^2 = 0,\end{aligned} \tag{24}$$

$$2\alpha a_1b_1b_0 - w^2a_1b_1b_0 - k^2a_0b_1^2 + w^2a_0b_1^2 + k^2a_1b_1b_0 + \alpha a_0b_1^2 - 3\beta a_0a_1^2 = 0, \tag{25}$$

$$\begin{aligned}&4k^2a_{-1}b_1^2 - \alpha a_1b_0^2 + 3\beta a_1a_0^2 - 4k^2a_1b_1b_{-1} - 2\alpha a_0b_1b_0 + k^2a_1b_0^2 - k^2a_0b_1b_0 + 4w^2a_1b_1b_{-1}\\&\quad - 4w^2a_{-1}b_1^2 - w^2a_1b_0^2 + 3\beta a_{-1}a_1^2 - 2\alpha a_1b_1b_{-1} - \alpha a_{-1}b_1^2 + w^2a_0b_1b_0 = 0,\end{aligned} \tag{26}$$

$$\begin{aligned}&k^2a_{-1}b_0^2 - 2\alpha a_{-1}b_1b_{-1} - 4k^2a_{-1}b_1b_{-1} - k^2a_0b_{-1}b_0 - 2\alpha a_0b_0b_{-1} - 4w^2a_1b_{-1}^2 + 4k^2a_1b_{-1}^2\\&\quad + 4w^2a_{-1}b_1b_{-1} + 3\beta a_1a_{-1}^2 + w^2a_0b_0b_{-1} + 3\beta a_0^2a_{-1} - \alpha a_{-1}b_0^2 - w^2a_{-1}b_0^2 - \alpha a_1b_{-1}^2 = 0,\end{aligned} \tag{27}$$

$$\alpha a_{-1}b_{-1}^2 - \beta a_{-1}^3 = 0, \tag{28}$$

$$2\alpha a_{-1}b_0b_{-1} - 3\beta a_0a_{-1}^2 - w^2a_{-1}b_0b_{-1} + w^2a_0b_{-1}^2 - k^2a_0b_{-1}^2 + \alpha a_0b_{-1}^2 + k^2a_{-1}b_0b_{-1} = 0. \tag{29}$$

Solving (16)–(29) simultaneously, we obtain

Case 1.

$$a_1 = -\frac{a_0b_1}{b_0}, \qquad a_{-1} = a_1 = b_0 = 0, \qquad b_1 = \frac{\beta a_0^2}{8\alpha b_{-1}}, \qquad k = \pm\sqrt{w^2+\alpha},$$

where a_0, b_0, b_{-1}, b_1 and w are free parameters, then the solutions of Eq. (7) are

$$u_{1,2}(x,t) = \frac{8\alpha a_0b_{-1}}{\beta a_0^2e^{(\pm\sqrt{w^2+\alpha}x+wt)} + 8\alpha b_{-1}^2e^{-(\pm\sqrt{w^2+\alpha}x+wt)}}. \tag{30}$$

If we set $\beta a_0^2 = 8\alpha b_{-1}^2$, the generality bell-type soliton solution can be obtained,

$$u'_{1,2}(x,t) = \frac{a_0}{2b_{-1}}\operatorname{sech}(\pm\sqrt{w^2+\alpha}x + wt), \tag{31}$$

where a_0, b_{-1}, and w are free parameters.

Case 2.

$$a_0=0,\qquad a_1=\mp\frac{b_0^2}{4b_{-1}}\sqrt{\frac{\alpha}{\beta}},\qquad a_{-1}=\pm\sqrt{\frac{\alpha}{\beta}}b_{-1},\qquad b_1=\frac{b_0^2}{4b_{-1}},\qquad k=\pm\sqrt{w^2-2\alpha},$$

where b_0, b_{-1} and w are free parameters, then the solutions of Eq. (7) are

$$u_{3,4}(x,t)=\sqrt{\frac{\alpha}{\beta}}\frac{[-b_0^2e^{\pm\sqrt{w^2-2\alpha}x+wt}+4b_{-1}^2e^{-(\pm\sqrt{w^2-2\alpha}x+wt)}]}{[b_0^2e^{\pm\sqrt{w^2-2\alpha}x+wt}+4b_{-1}b_0+4b_{-1}^2e^{-(\pm\sqrt{w^2-2\alpha}x+wt)}]}. \tag{32}$$

If we set $b_0^2=4b_{-1}^2$, the generality kink-type soliton solutions can be obtained,

$$u'_{3,4}(x,t)=-\sqrt{\frac{\alpha}{\beta}}\frac{[e^{(\pm\sqrt{w^2-2\alpha}x+wt)/2}-e^{-(\pm\sqrt{w^2-2\alpha}x+wt)/2}]}{[e^{(\pm\sqrt{w^2-2\alpha}x+wt)/2}+e^{-(\pm\sqrt{w^2-2\alpha}x+wt)/2}]}=-\sqrt{\frac{\alpha}{\beta}}\tanh\frac{\pm\sqrt{w^2-2\alpha}x+wt}{2}. \tag{33}$$

Case 3.

$$a_0=\pm b_0\sqrt{\frac{\alpha}{\beta}},\qquad b_1=\pm a_1\sqrt{\frac{\beta}{\alpha}},\qquad b_{-1}=\pm\sqrt{\frac{\beta}{\alpha}}a_{-1},\qquad k=\pm\sqrt{w^2-2\alpha},$$

where a_1, b_0, a_{-1} and w are free parameters, then the solutions of Eq. (4) are

$$u_{5,6}(x,t)=\frac{[a_1\sqrt{\alpha\beta}e^{(\pm\sqrt{w^2-2\alpha}x+wt)}+\alpha b_0+a_{-1}\sqrt{\alpha\beta}e^{-(\pm\sqrt{w^2-2\alpha}x+wt)}]}{[a_1\beta e^{(\pm\sqrt{w^2-2\alpha}x+wt)}+b_0\sqrt{\alpha\beta}+\beta a_{-1}e^{-(\pm\sqrt{w^2-2\alpha}x+wt)}]}, \tag{34}$$

which are new solitary wave solutions.

Case 4.

$$a_1=\frac{\alpha^2b_0^3\pm a_0^3\beta\sqrt{\alpha\beta}\mp\alpha a_0b_0^2\sqrt{\alpha\beta}-\alpha a_0^2b_0\beta}{4\alpha b_{-1}(b_0\sqrt{\alpha\beta}\mp a_0\beta)},\qquad a_{-1}=\pm\sqrt{\frac{\alpha}{\beta}}b_{-1}$$
$$b_1=-\frac{\alpha^2b_0^3\pm a_0^3\beta\sqrt{\alpha\beta}\mp\alpha a_0b_0^2\sqrt{\alpha\beta}-\alpha a_0^2b_0\beta}{4\alpha b_{-1}(a_0\sqrt{\alpha\beta}\mp b_0\alpha)},\qquad k=\pm\sqrt{w^2-2\alpha}$$

where a_0, b_0, b_{-1}, and w are free parameters, and then the solutions of Eq. (7) are

$$\begin{aligned}u_{7,8}(x,t)=&\sqrt{\frac{\alpha}{\beta}}[(\alpha a_0b_0^2\sqrt{\alpha\beta}+\alpha\beta a_0^2b_0-\alpha^2b_0^3-a_0^3\beta\sqrt{\alpha\beta})e^{\pm\sqrt{w^2-2\alpha}x+wt}\\&+4a_0b_{-1}b_0\alpha\sqrt{\alpha\beta}-4a_0^2b_{-1}\alpha\beta+(4\alpha^2b_0b_{-1}^2-4\alpha\sqrt{\alpha\beta}a_0b_{-1}^2)e^{-(\pm\sqrt{w^2-2\alpha}x+wt)}]\\&/[(\alpha^2b_0^3+a_0^3\beta\sqrt{\alpha\beta}-\alpha a_0b_0^2\sqrt{\alpha\beta}-\alpha\beta a_0^2b_0)e^{\pm\sqrt{w^2-2\alpha}x+wt}+4b_{-1}\alpha^2b_0^2\\&-4a_0b_0b_{-1}\alpha\sqrt{\alpha\beta}+(4b_0\alpha^2b_{-1}^2-4\alpha a_0b_{-1}^2\sqrt{\alpha\beta})e^{-(\pm\sqrt{w^2-2\alpha}x+wt)}],\end{aligned} \tag{35}$$

which are new solitary wave solutions.

4. Summary

In this paper, we have utilized the combined Exp-function method with variational iteration method to study the Klein–Gordon equation. As a result, some new explicit exact travelling wave solutions of the Klein–Gordon equation have been obtained which include generality bell-type soliton solution, generality kink-type soliton solutions and so on. As far as we know, some solutions are first found. From using the combined Exp-function method with variational iteration method, we found the method to be not only effective and convenient, but also the solutions, very general. This paper is a primary work of the combined Exp-function method with variational iteration method, we think that the method is worth further investigation in mathematics and physics.

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